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Application of prismatic constructions to gauge field theory

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ABSTRACT

We give a variational formula for the Chern–Simons invariants for a given bundle on a simplicial set with a connection using prismatic subdivision.

1. Introduction

Prismatic sets were introduced by Dupont and Ljungmann [8] in order to have the construction of an integration map in smooth Deligne cohomology. The author worked with the prismatic subdivision of a simplicial set S in connection with Lattice Gauge Theory [1]. In this paper, we give formulas for the difference of the Chern–Simons classes for a given bundle $F \to |S_{.}|$ with a connection ω using prismatic subdivision, where $|S_{.}|$ is the geometric realization of S. We do the variation of the space of connections for Chern–Simons classes using prismatic constructions. The reason for using prismatic construction is that we work on locally trivial bundles where the combinatorial methods for calculating invariants depend on the use of simplicial complexes. In general, a fibre is not a simplicial complex but one can have its natural decomposition into prisms. The development of the theory of bundles, trivializations and transition functions in the setting of simplicial sets leads us to Chern–Weil and Chern–Simons theory. We recall the prismatic constructions, simplicial bundle and transition functions from Akyar and Dupont [3]. The transition functions lead us to define the classifying map which plays an important role for the variational formula. We give the connection in the bundle over the prismatic subdivision of a simplicial set S using the classifying map. In this work we give the evaluation of the Chern–Simons form for the prismatic subdivision and the variational formula for the Chern–Simons classes for a given bundle $F \to |S_{.}|$ with a connection ω. This is our main result (Theorem 3.8) and this variation plays an important role in topological quantum field theory (see Witten [16]). Fixing one of the connections in Theorem 3.8 and considering the evaluation of the Chern–Simons form on each simplex in the prismatic subdivision can be used as the Lagrangian of a quantum field theory. We give an example at the end of this

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section. Furthermore we interpret the Chern–Simons functional in terms of simplicial forms and Deligne cohomology for a general case.

2. Characteristic classes and Chern–Simons theory

In this section, we refer to Dupont [6] for Chern–Weil Theory for a differentiable principal G-bundle $E \to M$, where G is a Lie group with its Lie algebra $\mathfrak{g}$ and M is a differentiable manifold. Let V be a finite dimensional real vector space and denote $S^k(V^*)$ by the vector space of symmetric multilinear real valued functions in k variables for $k \geqslant 1$. The set of differential forms on M with values in V is denoted by $\Omega^*(M,V)$.

Definition 2.1. 1) A linear map $P \in S^k(V^*)$, $P : V^{\otimes k} \to \mathbb{R}$, is called invariant under the symmetric group action induced by the adjoint representation. The set of invariant polynomials in $S^*(\mathfrak{g}^*)$ is denoted by $I^*(G)$, where $I^*(G) = \bigoplus_{k=0}^{\infty} I^k(G)$ is a subring of $S^*(\mathfrak{g}^*)$.

2) The corresponding cohomology class for $P \in I^k(G)$ is $w_E(P) \in H^{2k}(\Omega^*(M))$ which defines a multiplicative homomorphism

$$w_E : I^*(G) \to H\big(\Omega^*(M)\big)$$

and it is called the Chern–Weil homomorphism.

Let $\pi : E \to M$ be a principal G-bundle on a differentiable manifold M. Suppose ω is a connection in E with the associated curvature form $F_\omega \in \Omega^2(E,\mathfrak{g})$. We have ${F_\omega}^k = F_\omega \wedge \cdots \wedge F_\omega \in \Omega^{2k}(E,\mathfrak{g})$ for $k \geqslant 1$ so $P \in I^k(G)$ gives rise to the integrand $P(F_\omega^k) \in \Omega^{2k}(E)$. For $P \in I^*(G)$, $w_E(P)$ is called the characteristic class of E corresponding to P, that is, $w_E(P) = [P({F_\omega}^k)]$.

Definition 2.2 *(Simplicial form).* A simplicial n-form φ on a simplicial manifold $M = \{M_p\}$ is a sequence of n-forms $\varphi^{(p)}$ on $\Delta^p \times M_p$ such that $(\varepsilon^j \times \mathrm{id})^*\varphi^{(p)} = (\mathrm{id} \times d_j)^*\varphi^{(p-1)}$ on $\Delta^{p-1} \times M_p$, for all $j = 0,\dots,p$ and $p = 0,\dots,$ where Δ^p is the standard p-simplex, ε^j is the j-th face map and d_j is the j-th face operator.

Definition 2.3 *(Chern–Simons form).* Let $\pi : E \to M$ be a principal $SU(2)$-bundle and ω an $\mathfrak{su}(2)$-valued 1-form on E, where $\mathfrak{su}(2)$ is the Lie algebra of $SU(2)$. The real valued 4-form $\tilde{p}$ on E given by

$$\tilde{p} = \frac{1}{8\pi^2}\operatorname{Tr}(F_\omega \wedge F_\omega),$$

where the real-valued map $\operatorname{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is given by $\operatorname{Tr}(B,A) = \operatorname{Tr} BA$, is the lift of a unique 4-form p on M, that is, $\pi^*(p) = \tilde{p}$. The 3-form

$$TP(\omega) = \frac{1}{8\pi^2}\operatorname{Tr}\left(\omega \wedge F_\omega - \frac{1}{3}\omega \wedge \omega \wedge \omega\right)$$

is called the *Chern–Simons form.*

One can easily see that the form p in Definition 2.3 represents the second Chern class of the bundle $\xi(\omega)$, that is, $c_2(E) = \frac{1}{8\pi^2}\operatorname{Tr}(F_\omega \wedge F_\omega)$ (see Phillips and Stone [14]).

In general, TP is a real-valued invariant $(2k-1)$-form on the total space E (see Chern and Simons [5]).

In order to compute a related formula for the variational formula of the Chern–Simons classes for a given bundle $F \to |S_.|$ (see Akyar [2] for the definition of a bundle on a simplicial set) with a connection ω, we need the difference of Chern–Simons forms considered as a difference of differential characters. We study a certain graded ring $\hat{H}^*(M)$ for a smooth manifold M which is the ring of differential characters on M. If $\Lambda \subset \mathbb{R}$ is a proper subgroup, a differential character (mod Λ) is a homomorphism f from the group of smooth singular k-cycles to $\mathbb{R}/\Lambda$, whose coboundary is the mod Λ reduction of some (necessarily closed) differential form $\mathrm{w} \in \Omega^{k+1}(M)$. One can see that f uniquely determines not only w but a class $u \in H^{k+1}(M,\Lambda)$ whose real image is cohomologous to the de Rham class of w. One can recall the Chern–Simons form as a differential character using a lift of Weil homomorphism due to Cheeger and Simons [4] as follows:

Let $\xi = (E,M,\omega)$ be a principal G-bundle, where G is a Lie group. Let ${\Omega_{cl}}^*(M)$ denote the set of closed forms. The Chern–Weil homomorphism constructs a homomorphism $w : I^k(G) \to H^{2k}(BG,\mathbb{R})$, where BG is the classifying space of G and a natural transformations $W : I^k(G) \to {\Omega_{cl}}^{2k}(M)$ such that the following diagram of natural transformations commutes

$$\begin{tikzcd} I^*(G) \arrow[r, "w"] \arrow[d, "W"] & H^*(BG,\mathbb{R}) \arrow[d, "c_{\mathbb{R}}"] & H^*(BG,\Lambda) \arrow[l, "r"'] \arrow[d, "c_\Lambda"] \\ {\Omega_{cl}}^*(M) \arrow[r, "DR"] & H^*(M,\mathbb{R}) & H^*(M,\Lambda) \arrow[l, "r"'] \end{tikzcd} \tag{2.4}$$

where C_Λ, $C_\mathbb{R}$ are provided by the theory of characteristic classes and DR is the de Rham homomorphism, where Λ is a proper subring of $\mathbb{R}$. If $P \in I^k(G)$, $u \in H^*(BG,\Lambda)$ and F_ω is the curvature form of ξ, where $\xi \in \varepsilon(G)$ and $\varepsilon(G)$ is the category of the objects which are triples $\xi = (E, M, \omega)$ with morphisms being connection-preserving bundle maps, then $W(P) = P(\underbrace{F_\omega, \dots, F_\omega}_{k})$ and $C_\Lambda(u) = u(\xi)$ is the characteristic class. Set

$$K^{2k}(G,\Lambda) = \{(P,u) \in I^k(G) \times H^{2k}(BG,\Lambda) \mid w(P) = r(u)\},$$
$$R^k(M,\Lambda) = \{(\mathrm{w},u) \in {\Omega_{cl}}^k(M) \times H^k(M,\Lambda) \mid r(u) = [\mathrm{w}]\}$$

and $R^*(M,\Lambda) = \bigoplus R^k(M,\Lambda)$. Here $r : H^k(M,\Lambda) \to H^k(M,\mathbb{R})$ and $[\mathrm{w}]$ is the de Rham class of w and ${\Omega_{cl}}^k$ denotes the closed k-forms with periods lying in Λ. The diagram (2.4) induces a map $W \times C_\Lambda : K^*(G,\Lambda) \to R^*(M,\Lambda)$. On the other hand, we have a ring which is given by

$$\hat{H}^k(M,\mathbb{R}/\Lambda) = \{f \in \mathrm{Hom}(Z_k,\mathbb{R}/\Lambda) \mid f \circ \partial \in \Omega^{k+1}\}.$$

We set $\hat{H}^{-1}(M,\Lambda) = \Lambda$.

Definition 2.5 *(Differential character).* $\hat{H}^*(M,\mathbb{R}/\Lambda) = \bigoplus \hat{H}^k(M,\mathbb{R}/\Lambda)$ is a graded Λ-module whose objects are called *differential characters*.

There exists a unique natural transformation

$$S : K^*(G,\Lambda) \to \hat{H}^*(M,\mathbb{R}/\Lambda)$$

such that the diagram

$$\begin{tikzcd} & \hat{H}^*(M,\mathbb{R}/\Lambda) \arrow[d, "\delta_1,\delta_2"] \\ K^*(G,\Lambda) \arrow[ur, "S"] \arrow[r, "W \times C_\Lambda"'] & R^*(M,\Lambda) \end{tikzcd}$$

commutes and $S_{P,u} \in \hat{H}^{2k-1}(M,\mathbb{R}/\Lambda)$ is called the Chern–Simons class.

Theorem 2.6. *(Cheeger and Simons [4]) Let $(P,u) \in K^{2k}(G,\Lambda)$. For each $\xi \in \varepsilon(G)$, there exists a unique $S_{P,u} \in \hat{H}^{2k-1}(M,\mathbb{R}/\Lambda)$ satisfying*:

1) $\delta_1(S_{P,u}(\omega)) = P(F_\omega)$.
2) $\delta_2(S_{P,u}(\omega)) = u(\xi)$.
3) *If $\hat{\xi} \in \varepsilon(G)$ and $\phi : \xi \to \hat{\xi}$ is a morphism then $\phi^*(S_{P,u}(\hat{\omega})) = S_{P,u}(\omega)$.*

For a pair $([P],u)$, we have the Chern–Simons class for $\xi = (E, M, \omega)$ with ${F_\omega}^{k+1} = 0$ (see Dupont and Kamber [7]). We can give a more geometric interpretation of the Chern–Simons class.

Proposition 2.7. *Suppose that $M^{2k-1} = \partial W^{2k}$ is an oriented manifold and ξ extends to $\hat{\xi}$ over W. Let $\hat{\omega}$ be any extension of ω in $\hat{E}$. Setting $\hat{\xi} = \{\hat{E}, W, \hat{\omega}\}$ and we have the morphism $\xi \to \hat{\xi}$. Thus $S_{P,u}(\hat{\omega})|_{M^{2k-1}} = S_{P,u}(\omega)$. Moreover*

$$\langle S_{P,u}(\omega), [M]\rangle = \int_W P(F_{\hat{\omega}}) \bmod \mathbb{Z}.$$

Proof. We have

$$\begin{aligned}\langle S_{P,u}(\omega), [M]\rangle &= \langle \psi^*\bar{s}, [M]\rangle = \langle \psi^*\bar{s}, [\partial W]\rangle = \langle \delta\psi^*\bar{s}, [W]\rangle \\ &= \langle \mathfrak{I}([P(F_{\hat{\omega}})]), [W]\rangle = \int_W P(F_{\hat{\omega}}).\end{aligned}$$

Here $P(F_{\hat{\omega}})$ is the Chern–Simons form, $\mathfrak{I}$ is the integration map, $\psi : M \to \bar{E}/G = BG$ is the classifying map and $\psi^*\bar{s} \in C^{2k-1}(M,\mathbb{R}/\mathbb{Z})$ is called the Chern–Simons character. Moreover $\psi^*\bar{s} \bmod \mathbb{Z}$ is called the cohomology class. □

Corollary 2.8. *For two given connections ω_1 and ω_2 in a bundle $E \to M$, we have $\tilde{\omega} = (1-t)\omega_1 + t\omega_2$ in $M \times [0,1] = W^{2k}$ and the difference of the characters is the reduction of a form. In other words, the difference of the Chern–Simon classes on the manifold M is given as a variational formula*

$$\langle S_{P,u}(\omega_2), [M]\rangle - \langle S_{P,u}(\omega_1), [M]\rangle = \int_W P(F_{\tilde{\omega}}) = \int_M TP(\omega_1, \omega_2),$$

where $TP(\omega_1, \omega_2) = \int_0^1 i_{d/dt} P(F_{\tilde{\omega}})\, dt$, where $i_{d/dt}$ is the usual interior product in the t-variable.

Example 2.9. Suppose $E = M \times G$ and $M = M^{2k-1}$ is a closed, oriented manifold, $P \in I^k(G)$. Take $W = M \times [0,1]$ and for any connection ω in E and ω_{MC} we write

$$\tilde{\omega} = (1-t)\omega_{MC} + t\omega = \omega_{MC} + tA$$

which is a connection in $\tilde{E} = W \times G$, where $A = \omega - \omega_{MC}$. We get

$$\langle S_{P,u}(\omega), [M]\rangle - \langle S_{P,u}(\omega_{MC}), [M]\rangle = \int_{M\times[0,1]} P(F_{\tilde{\omega}}),$$

since $S_{P,u}(\omega_{MC}) = 0$ we have

$$\langle S_{P,u}(\omega), [M]\rangle = \int_M TP(A),$$

where $TP(A) \overset{\mathrm{def}}{=} \int_0^1 i_{d/dt} P(F_{\tilde{\omega}})\, dt$ and $TP(A)$ is an algebraic expression in A.

3. Applications to gauge field theory

In this section, we evaluate the Chern–Simons form on a prismatic set for a given bundle $F \to |S_{\cdot}|$ with a connection ω using the prismatic subdivision of the simplicial set S. For clarification we recall some notations and explanations from Akyar and Dupont [3]:

Definition 3.1. Given $p \geqslant 0$, a $(p+1)$-multi-simplicial set S is a sequence $\{S_{q_0,\ldots,q_p}\}$ which is a simplicial set in each variable q_j, $j = 0, \ldots, p$ and such that the face and degeneracy operators

$$d_j^k : S_{q_0,\ldots,q_p} \to S_{q_0,\ldots,q_k-1,\ldots,q_p},$$
$$s_j^k : S_{q_0,\ldots,q_p} \to S_{q_0,\ldots,q_k+1,\ldots,q_p}$$

for $k = 0, \ldots, p$, satisfying the simplicial identities and so that they commute with d_m^l, s_m^l for $k \neq l$, where $l = 0, \ldots, p$.

Definition 3.2 *(Prismatic set).* A prismatic set PS is a sequence $\{P_pS\} = \{P_pS_{q_0,\ldots,q_p}\}$ of $(p+1)$-multi-simplicial sets together with face operators $d_k : P_pS_{q_0,\ldots,q_p} \to P_{p-1}S_{q_0,\ldots,\hat{q}_k,\ldots,q_p}$ commuting with d_m^l and s_m^l (saying $d_m^k = s_m^k = \mathrm{id}$ on the right) such that $\{P_pS\}$ is a Δ-set. Explicitly,

$$P_pS_{q_0,\ldots,q_p} := S_{q_0+\cdots+q_p+p},$$

where $q_0 + \cdots + q_p = q$, is the $(p+1)$-prismatic set with face operators

$$d_j^i : P_pS_{q_0,\ldots,q_i,\ldots,q_p} = S_{q+p} \to P_pS_{q_0,\ldots,q_i-1,\ldots,q_p} = S_{q+p-1}$$

which are defined by

$$d_j^i := d_{q_0+\cdots+q_{i-1}+i+j},$$

$j = 0, \ldots, q_i$, $i = 0, \ldots, p$, and degeneracy operators

$$s_j^i : P_pS_{q_0,\ldots,q_i,\ldots,q_p} = S_{q+p} \to P_pS_{q_0,\ldots,q_i+1,\ldots,q_p} = S_{q+p+1}$$

which are defined by

$$s_j^i := s_{q_0+\cdots+q_{i-1}+i+j},$$

$j = 0, \ldots, q_i$, $i = 0, \ldots, p$. The face maps $d_i : P_pS_{q_0,\ldots,q_p} \to P_{p-1}S_{q_0,\ldots,\hat{q}_i,\ldots,q_p}$ are the operators corresponding to the inclusions $\Delta^{q_0+\cdots+\hat{q}_i+\cdots+q_p+p-1} \to \Delta^{q_0+\cdots+\cdots+q_p+p}$.

Moreover if similarly there are given degeneracy operators $s_k : P_pS_{q_0,\ldots,q_p} \to P_{p+1}S_{q_0,\ldots,q_k,q_k,\ldots,q_p}$ making $\{P_pS\}$ a simplicial set, a prismatic set PS is called a strong prismatic set.

Example 3.3 *(Triangulated fibre bundles).* Given a smooth fibre bundle $\pi : Y \to Z$ with $\dim Y = m+n$, $\dim Z = m$ and compact fibres possibly with boundary. By a theorem of Johnson [13], there are smooth triangulations K and L of Y and Z, respectively and a simplicial map $\pi' : K \to L$ in the following commutative diagram

$$\begin{tikzcd} |K| \arrow[r, "\approx"] \arrow[d, "|\pi'|"'] & Y \arrow[d, "\pi"] \\ |L| \arrow[r, "\approx"] & Z \end{tikzcd}$$

and the horizontal maps are homeomorphisms which are smooth on each simplex. Let K be an ordered simplicial complex as in Dwyer and Henn [12, Section 3] and let $|K| = \bigsqcup_{\tau \in K_k} \Delta^k \times \tau / \sim$, $k = 0, \ldots, \dim K$, be the geometric realization.

A simplex τ in K has vertices $\tau = (b_0^0, \ldots, b_{q_0}^0 | \cdots | b_0^p, \ldots, b_{q_p}^p)$ with $\sigma = (a_0, \ldots, a_p)$ in L such that $\pi'(b_j^i) = a_i$. So geometrically, for an open simplex $\mathring{\sigma}$ in L, we have

$$\pi^{-1}\big(|\mathring{\sigma}|\big) \approx |\mathring{\sigma}| \times \bigsqcup_{\tau \in \pi^{-1}(\sigma)} \Delta^{q_0 \cdots q_p} \times \tau$$

where $\Delta^{q_0 \cdots q_p} = \Delta^{q_0} \times \cdots \times \Delta^{q_p}$.

In order to see how the prismatic subdivision appears, we recall the following example from Akyar and Dupont [3].

Example 3.4 *(Prismatic triangulation of a simplicial set).* Let $f : S \to \{*\}$ be a simplicial map of simplicial sets, where $\{*\}$ is the simplicial set with one element in each degree and

$$P_p(f)_{q_0,\ldots,q_p} = \big\{(\bar{\sigma}, \sigma) \in \{*\} \times S_{q_0+\cdots+q_p+p} \;\big|\; f(\sigma) = \mu_{q_0,\ldots,q_p}(\bar{\sigma})\big\}$$

where $\mu_{q_0,\ldots,q_p} = \hat{s}_{q+p} \circ s_{q_0+\cdots+q_p+p-1} \cdots s_{q_0+\cdots+q_{p-1}+p} \circ \cdots \circ \hat{s}_{q_0} \circ s_{q_0-1} \cdots s_0$, where "ˆ" means that the term is left out. We call $P_p(f) = P_pS$ the p-th *prismatic subdivision* of S.

We have for each p the geometric realization

$$|P_pS_\cdot| = \bigsqcup_{q_0 \cdots q_p} \Delta^{q_0 \cdots q_p} \times S_{q_0+\cdots+q_p+p}/\sim$$

with equivalence realization "$\sim$" generated by the face and degeneracy maps

$$\varepsilon_j^i : \Delta^{q_0 \cdots q_i \cdots q_p} \to \Delta^{q_0 \cdots q_i+1 \cdots q_p} \quad \text{and}$$
$$\eta_j^i : \Delta^{q_0 \cdots q_i \cdots q_p} \to \Delta^{q_0 \cdots q_i-1 \cdots q_p},$$

respectively. The sequences of Δ-spaces $\{|P_\cdot S_\cdot|\}$ give the fat realization:

$$\big\||P_\cdot S_\cdot|\big\| = \bigsqcup_{p \geqslant 0} \Delta^p \times |P_pS_\cdot|/\sim,$$

with equivalence relation "$\sim$" generated by the face operators $|d_i| : |P_pS_\cdot| \to |P_{p-1}S_\cdot|$ which are given by $|d_i| = \pi_i \times d_i$ with the natural projections $\pi_i : \Delta^{q_0 \cdots q_p} \to \Delta^{q_0 \cdots \hat{q}_i \cdots q_p}$, where $i = 0, \ldots, p$. For each $t \in \mathring{\Delta}^p$, the map $|P_pS_\cdot| \to |S_\cdot|$ is a homeomorphism. In particular, $\||P_\cdot S_\cdot|\| \to |S_\cdot|$ is a homotopy equivalence.

Example 3.5 *(Nerve of coverings).* (Dupont and Ljungmann [8]) Given a covering $\mathcal{U} = \{U_i\}$ of Z, there is a covering $\mathcal{W} = \{W_i = \pi^{-1}(U_i)\}$ of Y, and for each i, $\mathcal{V}^i$ is an open cover of W_i giving a covering $\mathcal{V} = \bigcup \mathcal{V}^i$ of Y. Then $P_pN(\mathcal{V}/\mathcal{U})_{q_0,\ldots,q_p} = \bigsqcup V_{j_0^0}{}^{i_0} \cap \cdots \cap V_{j_{q_0}^0}{}^{i_0} \cap \cdots \cap V_{j_{q_p}^p}{}^{i_p}$, where $V_j^i \in \mathcal{V}^i$, is given with face and degeneracy maps as inclusions.

Note. Here $\mathcal{U} = \{U_i = \mathrm{st}(a_i)\}$ where $a_i \in L^0$ is a 0-simplex in L and $\mathcal{V}^i = \{V_j^i = \mathrm{st}(b_j^i)\}$, where $b_j^i \in \pi^{-1}(a_i) \cup K^0$.

Let us recall the transition functions from Akyar [2] for a given bundle over the realization of a simplicial set S and $\sigma \in S_p$. Given a bundle $F \to |S_\cdot|$ and a set of trivializations, we get for each face τ of say $\dim \tau = q < p = \dim \sigma$ in σ, a *transition function* $\nu_{\sigma,\tau} : \Delta^q \to G$ as follows: The bundle map Θ given by the diagram

$$\begin{array}{ccc} \Delta^q \times (d_{i_p\cdots i_0}\sigma) \times G & \xrightarrow{\Theta} & \Delta^p \times (\sigma) \times G \\ \downarrow & & \downarrow \\ \Delta^q \times d_{i_p\cdots i_0}\sigma & \xrightarrow{\varepsilon^{i_0\cdots i_p}} & \Delta^p \times \sigma \end{array}$$

where $d_i\sigma = \tau$, $\Theta = \varphi_\sigma \circ \bar{\varepsilon}^{i_0\cdots i_p} \circ \varphi^{-1}_{d_{i_p\cdots i_0}\sigma}$, $\bar{\varepsilon}^{i_0\cdots i_p} = \bar{\varepsilon}^{i_0} \circ \cdots \circ \bar{\varepsilon}^{i_p}$, $\varepsilon^{i_0\cdots i_p} = \varepsilon^{i_0} \circ \cdots \circ \varepsilon^{i_p}$ and $d_{i_p\cdots i_0} = d_{i_p} \circ \cdots \circ d_{i_0}$, determines $\nu_{\sigma,\tau}$ by the formula

$$\Theta(t, g) = \big(\varepsilon^{i_0} \circ \cdots \circ \varepsilon^{i_p}(t), \nu_{\sigma,\tau}(t)g\big), \quad t \in \Delta^q,\ g \in G.$$

Let $\bar{P}_p S_{q_0,\ldots,q_p} := S_{q_0+\cdots+q_p+2p+1}$ be another prismatic set for a simplicial set S given with face and degeneracy operators inherited from the ones of S_{q+2p+1} (see Akyar and Dupont [3]). There is a map $h : \||\bar{P}_{\bullet}S_{\bullet}|\| \to \||P_{\bullet}S_{\bullet}|\|$ defined by $h(t, s^0, \ldots, s^p, x) = (t, s^0, \ldots, s^p, d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots \circ d_{q+2p+1}x)$, $x \in S_{q+2p+1}$, where $q = q_0 + \cdots + q_p$ and $\||\bar{P}_{\bullet}S_{\bullet}|\| = \bigsqcup_{p\geqslant 0} \Delta^p \times \Delta^{q_0\cdots q_p} \times \bar{P}_p S_{q_0,\ldots,q_p}/\sim$ is defined with the equivalence relation given similarly as described for $\||P_{\bullet}S_{\bullet}|\|$.

The transition functions are used to define the classifying map $\bar{m} : \||\bar{P}_{\bullet}S_{\bullet}|\| \to BG$ (see also Akyar and Dupont [3]), the map $\tilde{m} : \bar{F} \to EG$ and a connection $\tilde{\omega}$ in $\bar{F} \to \||\bar{P}_{\bullet}S_{\bullet}|\|$, where $\bar{F} \cong \bar{m}^*(EG)$.

We assume that we already have a connection $\tilde{\omega}'$ in the bundle $\bar{F} \to \||\bar{P}_{\bullet}S_{\bullet}|\|$ and the other one $\tilde{\omega}$ can be defined as

$$\tilde{\omega} = \tilde{m}^*(\omega_\gamma) = \tilde{m}^*\left(\sum_{i=0}^{p} t_i' g_i^{-1}(t)\, dg_i(t)\right),$$

where ω_γ is the universal connection in the universal bundle $EG \to BG$. The bundle over $\||\bar{P}_{\bullet}S_{\bullet}|\|$ enables us to find the variational formula of the Chern–Simons classes for the bundle over $|S_{\bullet}|$.

Now, we compare these two connections $\tilde{\omega}$, $\tilde{\omega}'$ in $\bar{F} \to \||\bar{P}_{\bullet}S_{\bullet}|\|$ so that we give the variational formula of the Chern–Simons class for $F \to |S_{\bullet}|$ with $\tilde{\omega}$ and $\tilde{\omega}'$ as a difference form.

Definition 3.6 *(The Chern–Simons functional).* The Chern–Simons form on each simplex x of $\||\bar{P}_{\bullet}S_{\bullet}|\|$ is

$$\big\langle S_{P,u}(\tilde{\omega}), \big[\big\||\bar{P}_{\bullet}S_{\bullet}|\big\|\big]\big\rangle_x = \int_{\Delta^p \times \Delta^{q_0\cdots q_p}} P(F_{\tilde{\omega}}). \tag{3.7}$$

We can conclude the main result as the following theorem.

Theorem 3.8. *For two connections $\tilde{\omega}$ and $\tilde{\omega}'$ in a bundle $\bar{F} \to \||\bar{P}_{\bullet}S_{\bullet}|\|$, the variational formula of the Chern–Simons classes on each simplex x of $\||\bar{P}_{\bullet}S_{\bullet}|\|$ is given as a difference form*

$$\begin{aligned} \big\langle S_{P,u}(\tilde{\omega}'), \big[\big\||\bar{P}_{\bullet}S_{\bullet}|\big\|\big]\big\rangle_x - \big\langle S_{P,u}(\tilde{\omega}), \big[\big\||\bar{P}_{\bullet}S_{\bullet}|\big\|\big]\big\rangle_x &= \int_{I\times\Delta^p\times\Delta^{q_0\cdots q_p}} P(F_{\bar{\omega}}) \\ &= \int_{\partial(I\times\Delta^p\times\Delta^{q_0\cdots q_p})} TP(\tilde{\omega}, \tilde{\omega}') \end{aligned} \tag{3.9}$$

where $\bar{\omega} = (1-t)\tilde{\omega} + t\tilde{\omega}'$ and $TP(\tilde{\omega}, \tilde{\omega}') = \int_0^1 i_{d/dt} P(F_{\bar{\omega}})\, dt$.

Example 3.10. Let us consider two different compatible transition functions corresponding to parallel transport functions by varying with respect to the parameter t in Δ^p, saying that $t' = t + \lambda t$. Varying t defines two different classifying maps which also determine two connections $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in the bundle $\bar{F} \to \||\bar{P}_{\bullet}S_{\bullet}|\|$. The variational formula is given by

$$\big\langle S_{P,u}(\tilde{\omega}_1), \big[\big\||\bar{P}_{\bullet}S_{\bullet}|\big\|\big]\big\rangle - \big\langle S_{P,u}(\tilde{\omega}_2), \big[\big\||\bar{P}_{\bullet}S_{\bullet}|\big\|\big]\big\rangle = \int_{I\times\Delta^p\times\Delta^{q_0\cdots q_p}} P(F_{\bar{\omega}})$$

where $\bar{\omega} = (1-t)\tilde{\omega}_1 + t\tilde{\omega}_2$, as the variation of the connection for the Chern–Simons classes.

Note. Construction of a prismatic set which corresponds to the nerve of the covering by stars of vertices and a classifying map on it gives us a principal G-bundle with a connection and explicit formulas for characteristic classes via the usual Chern–Weil and Chern–Simons theory.

On the other hand, the Chern–Simons functional $S_{P,u}(\omega)$ can be used as the Lagrangian of a quantum field theory by fixing one of the connections, let us call it ω_0 and this leads us to path integrals $Z(M) = \int e^{2\pi i S_{P,u}(\omega)}\, d\omega$ over the space of all connections on the 3-manifold (see Freed [9], Huebschmann [10], Witten [16], Dijkgraaf and Witten [11], Rabin [15]).

Theorem 3.8 can be incorporated into the paper by Dupont and Ljungmann [8]. If one wants to have a better formula (see Theorem 1.1 in [8]) one can use Deligne cohomology which leads a combinatorial version of the Chern–Simons function. In other words, Theorem 3.8 is a way to avoid the use of Deligne cohomology.

Note. When $\bar{F} \to \||\bar{P}_{\cdot}S_{\cdot}|\|$ is topologically trivial, one usually takes $\tilde{\omega}$ to be the trivial connection and $S_{P,u}(\tilde{\omega})$ to be zero; for a flat connection $\tilde{\omega}$, the forms $TP(\tilde{\omega}, \tilde{\omega}')$ are then closed and $S_{P,u}(\tilde{\omega}')$ calculates the customary Chern–Simons invariants of the flat connection $\tilde{\omega}'$.

4. Another approach to integration with Deligne cohomology

The present work can be incorporated into the paper by Dupont and Ljungmann [8]. Now we can interpret $\int_{\Delta^p \times \Delta^{q_0 \cdots q_p}} P(F_{\tilde{\omega}})$ given by (3.7) in terms of integration of simplicial forms and Deligne cohomology (in order to have a combinatorial version for (3.9)). If we fix $\tilde{\omega}$ and only consider the first term in (3.9) we get a better formula for the Chern–Simons form on each simplex of $\||\bar{P}_{\cdot}S_{\cdot}|\|$. Let us recall some facts from [8].

Theorem 4.1. *(Dupont and Ljungmann [8]) Given a fiber bundle $\pi : Y \to Z$ with compact, oriented n-dimensional fibers and suitable coverings $\mathcal{V}$ and $\mathcal{U}$ of Y and Z, respectively. Then there is a map*

$$\int_{[Y/Z]} : \Omega^{*+n}\big(|N\mathcal{V}|\big) \to \Omega^{*}\big(|N\mathcal{U}|\big),$$

where $N\mathcal{V}$, $N\mathcal{U}$ denote the nerves of the coverings $\mathcal{V}$, $\mathcal{U}$, respectively, that is, given an open cover $\mathcal{U} = \{U_i\}$ of Z, the nerve $N\mathcal{U} = \{N\mathcal{U}(p)\}$ of the covering is given by $N\mathcal{U}(p) = \bigsqcup_{i_0,\ldots,i_p} U_{i_0} \cap \cdots \cap U_{i_p}$, here $U_{i_0} \cap \cdots \cap U_{i_p} = U_{i_0 \cdots i_p}$. $N\mathcal{U}$ is a simplicial manifold with the face $d_j : U_{i_0 \cdots i_p} \to U_{i_0 \cdots \hat{i}_j \cdots i_p}$ and degeneracy maps $s_j : U_{i_0 \cdots i_p} \to U_{i_0 \cdots i_j i_j \cdots i_p}$ given as inclusions.

Definition 4.2. A simplicial n-form $\mathrm{w} = \{\mathrm{w}^{(p)}\}$ on $N\mathcal{U}$ can be given in a similar way as in Definition 2.2 considering M as $N\mathcal{U}$.

Definition 4.3 *(Prismatic forms).* A prismatic n-form is a collection $\mathrm{w} = \{\mathrm{w}_{q_0 \cdots q_p}\}$ of forms $\mathrm{w}_{q_0 \cdots q_p} \in \Omega^n(\Delta^p \times \Delta^{q_0 \cdots q_p} \times P_p N(\mathcal{V}/\mathcal{U})_{q_0 \cdots q_p})$ satisfying the equivalence relations (see Definition 3.3 in Dupont and Ljungmann [8]). A form is called normal if it also satisfies the equivalence relations for the degeneracy maps.

The complex of normal prismatic form is $\Omega^*(|PN\mathcal{V}/\mathcal{U}|)$ and we have

$$\Omega^*\big(|PN\mathcal{V}/\mathcal{U}|\big) = \bigoplus_{p+q+r=n} \Omega^{p,q,r}\big(|PN\mathcal{V}/\mathcal{U}|\big),$$

where $q = q_0 + \cdots + q_p$ and $\Omega^{p,q,r}(|PN\mathcal{V}/\mathcal{U}|)$ is the set of forms of degree p in the barycentric coordinates of the first simplex, of degree q_0 in the second and so on and finally of degree r in some local coordinates on the nerve of the covering. Thus $\Omega^*(|PN\mathcal{V}/\mathcal{U}|)$ becomes a triple-complex.

Given a triangulation L of a smooth m-manifold Z there is an open cover $\mathcal{U}$ given by the stars $\mathrm{st}(a)$, where $a \in L^0$ and $\forall \sigma \in L$, $\overline{\mathrm{st}(\sigma)} = \bigcup_{\tau \in L^m, \sigma \subseteq \tau} |\tau|$ is the closed star which inherits a natural triangulation L_σ from L. This gives $|L_\sigma| \cong \overline{\mathrm{st}(\sigma)}$.

Definition 4.4. i) The triangulated nerve NL is the simplicial complex with p-simplices given by $N_p L = \bigsqcup_{\sigma \in L^p} |L_\sigma|$ for $\sigma = (a_0, \ldots, a_p)$ the face and degeneracy operators d_j, s_j are inclusions.

ii) A p-simplicial form w on $|NL|$ is a collection of forms $\mathrm{w} = \{\mathrm{w}^{(p)}\}$ living on $\bigsqcup_{\sigma \in L^p} \bigsqcup_i \Delta^p \times \Delta^i \times L_\sigma^{(i)}$, where $L_\sigma^{(i)}$ is the discrete set of i-simplices in L_σ.

We recall an integration map $\int_{[Y/Z]} : \Omega^{*+n}(|NK|) \to \Omega^*(|NL|)$ from Dupont and Ljungmann [8]. For a simplex $\sigma = (a_0, \ldots, a_p) \in L$ we have a chain map

$$AW : P_m C_k(K/L_\sigma) \to \bigoplus_{k_1+k_2=k} P_p C_{k_1}(K/\sigma) \otimes P_m C_{k_2}(K/L_\sigma)$$

by

$$AW(\tau) = \sum_{0 \leqslant q_j \leqslant q_{i_j}} \tau^{q_0 \cdots q_p} \otimes \tau_{q_0 \cdots q_p},$$

where

$$\tau^{q_0\cdots q_p} = \big[b_0^{i_0},\dots,b_{q_0}^{i_0}\big|\cdots\big|b_0^{i_p},\dots,b_{q_p}^{i_p}\big]$$

and

$$\tau_{q_0\cdots q_p} = \big[b_0^{0},\dots,b_{q_0}^{0}\big|\cdots\big|b_{q_j}^{i_j},\dots,b_{q_{i_j}}^{i_j}\big|\cdots\big|b_0^{m},\dots,b_{q_m}^{m}\big],$$

for $0 \leqslant q_j \leqslant q_{i_j}$, where $i, j = 0,\dots,p$. Here $P_pC_{q_0\cdots q_p}(K/L)$ is the free abelian group generated by $P_pS(K/L)_{q_0\cdots q_p}$ and $P_pS(K/L)_{q_0\cdots q_p} \subseteq S_{p+q_0+\cdots+q_p} \times S_p(L)$ is the subset of pairs of simplices (τ,η) so that q_i+1 of the vertices in τ lie over the i-th vertex in η. Let $(\tau,\eta) \in P_mC_K(K/L_\sigma)$ then since η is a top-dimensional simplex in L_σ, we have $\sigma \subseteq \eta$. Let $i_0,\dots,i_p \in \{0,\dots,m\}$ denote the indices of the corresponding vertices of σ in η. Write τ as $\tau = (b_0^0,\dots,b_{q_0}^0|\cdots|b_0^m,\dots,b_{q_m}^m)$, where the i-th block $|b_0^i,\dots,b_{q_i}^i|$ lies over the i-th vertex in η. Then the integration map is given by

$$\int_{K/L} \mathrm{w}_{|_{\Delta^p\times\eta}} = \sum_{\tau\in P_mS_n(K/\eta)} \sum_{0\leqslant q_j\leqslant q_{i_j}} \varepsilon(\tau) \int_{\Delta^{p+q}\times\tau_{q_0\cdots q_p}/\Delta^p\times\eta} \mathrm{w}^{(p+q)}_{\tau^{q_0\cdots q_p}},$$

where $\mathrm{w}^{(p+q)}_{\tau^{q_0\cdots q_p}} \in \Omega^{*+n}(\Delta^p \times K_{\tau^{q_0\cdots q_p}})$ and $\varepsilon(\tau)$ is the sign of τ in $[Y_{|_\eta}]$ and $q = \sum_{i=0}^p q_i$. Here w is restricted to $\Delta^{p+q} \times \tau_{q_0\cdots q_p}$ and it is integrated along the fiber over $\Delta^p \times \eta$ with respect to the map $\Delta^{p+q} \to \Delta^p$ given by

$$(t_0,\dots,t_{p+q}) \to \left(\sum_{i=0}^{q_0} t_i, \sum_{i=q_0+1}^{q_0+q_1+1} t_i, \dots, \sum_{i=q_0+\cdots+q_{p-1}+p}^{q_0+\cdots+q_p+p} t_i\right)$$

and the map $\tau_{q_0\cdots q_p} \to \eta$ is the restriction of π.

For $\mathrm{w} \in \Omega^{n+k}(|N\mathcal{V}|)$

$$\left(\int_{[Y/Z]} \mathrm{w}\right)_{|_{\Delta^p\times U_{i_0\cdots i_p}}} = \int_{\Delta^p\times W_{i_0\cdots i_p}/\Delta^p\times U_{i_0\cdots i_p}} \tilde{\Phi}^* f^*\mathrm{w}$$

where the right hand side denotes usual integration along the fibers, $f : |PN\mathcal{V}/\mathcal{U}| \to |N\mathcal{V}|$ and $\tilde{\Phi} : |N\mathcal{W}| \to |PN\mathcal{V}/\mathcal{U}|$. This integration enables us to give a better formulation for (3.9) when $\tilde{\omega}$ is fixed.

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